

# Sharp estimates on the tail behavior of a multistable distribution

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## Abstract

Multistable distributions are natural extensions of symmetric  $\alpha$  stable distributions. They have been introduced quite recently by Falconer, Lévy Véhel and their co-authors in [1, 2, 3]. Roughly speaking such a distribution is obtained by replacing the constant parameter  $\alpha$  of a symmetric stable distribution by a (Lebesgue) measurable function  $\alpha(x)$  with values in  $[a, 2]$ , where  $a > 0$  denotes a fixed arbitrarily small real number.

Let  $Y$  be an arbitrary symmetric  $\alpha$  stable random variable of scale parameter  $\sigma > 0$ , an important classical result concerning the heavy-tailed behavior of its distribution (see e.g. [4]) is that there exists an explicit constant  $C(\alpha) > 0$ , only depending on  $\alpha \in (0, 2)$ , such that  $\lim_{\lambda \rightarrow +\infty} (C(\alpha)\sigma^\alpha \lambda^{-\alpha})^{-1} \mathbb{P}(|Y| > \lambda) = 1$ . In this article, we show that the latter result can be extended to the setting of multistable random variables, when the function  $\alpha(x)$  is with values in an arbitray compact interval  $[a, b]$  contained in  $(0, 2)$ .

## 1 Introduction and statement of the main result

Let us first briefly recall the definition of the  $\alpha(x)$ -multistable stochastic integral which was quite recently introduced in [3]. To this end, we need to fix some definitions and notations. We denote by  $a > 0$  a fixed arbitrarily small real number and we denote by  $\alpha : \mathbb{R} \rightarrow [a, 2]$  an arbitrary fixed Lebesgue measurable function;  $\mathcal{F}_\alpha$ , the corresponding variable exponent Lebesgue space, is defined as,

$$\mathcal{F}_\alpha = \left\{ f : f \text{ is Lebesgue measurable with } \int_{\mathbb{R}} |f(x)|^{\alpha(x)} dx < \infty \right\}.$$

The space  $\mathcal{F}_\alpha$  is equipped with a quasinorm denoted by  $\|\cdot\|_\alpha$ ; for every  $f \in \mathcal{F}_\alpha^* = \mathcal{F}_\alpha \setminus \{0\}$ ,  $\|f\|_\alpha$  is defined as the unique  $\lambda_{0,f} \in (0, +\infty)$ , such that,  $\int_{\mathbb{R}} |\lambda_{0,f}^{-1} f(x)|^{\alpha(x)} dx = 1$ , moreover

one sets  $\|0\|_\alpha = 0$ . Let us recall that to say that  $\|\cdot\|_\alpha$  is a quasinorm means that  $\|\cdot\|_\alpha$  satisfies the following 3 properties:

- for all  $f \in \mathcal{F}_\alpha$ , one has  $f = 0$  if and only if  $\|f\|_\alpha = 0$ ;
- for all  $f \in \mathcal{F}_\alpha$  and  $\delta \in \mathbb{R}$ , one has  $\|\delta f\|_\alpha = |\delta| \|f\|_\alpha$ ;
- there is a constant  $k > 0$ , such that for all  $f, g \in \mathcal{F}_\alpha$ , one has  $\|f + g\|_\alpha \leq k(\|f\|_\alpha + \|g\|_\alpha)$  (weak triangle inequality).

The following theorem is an important result of [3], which allows to define on  $\mathcal{F}_\alpha$  the multi-stable stochastic integral; it has been obtained thanks to Kolmogorov's extension Theorem.

**Theorem 1.1** [3] *There exists a real-valued stochastic process indexed by the space  $\mathcal{F}_\alpha$ , denoted by  $\{I(f) : f \in \mathcal{F}_\alpha\}$ , whose finite dimensional distributions are characterized by the following property: for all integer  $d \geq 1$  and all  $f_1, \dots, f_d \in \mathcal{F}_\alpha$ ,  $\Phi_{I(f_1), \dots, I(f_d)} = \Phi_{f_1, \dots, f_d}$  the characteristic function of the random vector  $(I(f_1), \dots, I(f_d))$ , satisfies for all  $(\theta_1, \dots, \theta_d) \in \mathbb{R}^d$ ,*

$$\Phi_{f_1, \dots, f_d}(\theta_1, \dots, \theta_d) = \exp \left\{ - \int_{\mathbb{R}} \left| \sum_{l=1}^d \theta_l f_l(x) \right|^{\alpha(x)} dx \right\}. \quad (1.1)$$

Recall that, generally speaking, the distribution of an arbitrary random vector  $(X_1, \dots, X_d)$  is completely determined by  $\Phi_{X_1, \dots, X_d}$  its characteristic function, which is defined for all  $(\theta_1, \dots, \theta_d) \in \mathbb{R}^d$ , as,

$$\Phi_{X_1, \dots, X_d}(\theta_1, \dots, \theta_d) = \mathbb{E} \left( \exp i \sum_{l=1}^d \theta_l X_l \right). \quad (1.2)$$

For each  $f \in \mathcal{F}_\alpha$ ,  $I(f)$  is called an  $\alpha(x)$ -multistable random variable and its distribution is called an  $\alpha(x)$ -multistable distribution. Generally speaking, in many applied and theoretical problems, it is important to have a sharp estimates on the tail behavior of a probability distribution. The following theorem, which is our main result, provides such an estimation in the case of an  $\alpha(x)$ -multistable distribution.

**Theorem 1.2** *Assume that there is  $b \in (a, 2)$  such that for almost all  $x \in \mathbb{R}$ ,  $\alpha(x) \in [a, b]$ . Let  $C$  be the continuous strictly positive function defined for all  $\gamma \in [a, b]$  as,*

$$C(\gamma) = \frac{2}{\pi} \text{ if } \gamma = 1, \text{ and } C(\gamma) = \frac{1 - \gamma}{\Gamma(2 - \gamma) \cos(2^{-1}\pi\gamma)} \text{ else,} \quad (1.3)$$

where  $\Gamma$  is the usual "Gamma" function. For each  $f \in \mathcal{F}_\alpha$  and real number  $\lambda > 0$ , let us set,

$$T_f(\lambda) = \int_{\mathbb{R}} |\lambda^{-1} f(x)|^{\alpha(x)} C(\alpha(x)) dx. \quad (1.4)$$

Then, one has,

$$\lim_{\lambda \rightarrow +\infty} \left\{ \sup_{f \in \mathcal{S}_\alpha} \left| \frac{\mathbb{P}(|I(f)| > \lambda)}{T_f(\lambda)} - 1 \right| \right\} = 0, \quad (1.5)$$

where  $\mathcal{S}_\alpha = \{f \in \mathcal{F}_\alpha : \|f\|_\alpha = 1\}$  denotes the unit sphere of  $\mathcal{F}_\alpha$ .

Before ending this introduction, let us make some remarks concerning Theorem 1.2.

**Remarks:**

- Theorem 1.2 is an extension to the setting of multistable random variables of Property 1.2.15, on page 16 in [4]. Indeed, assuming that for almost all  $x \in \mathbb{R}$ ,  $\alpha(x) = \alpha$  where  $\alpha \in (0, 2)$  is a constant, then  $I(f)$  reduces to a usual symmetric  $\alpha$  stable random variable of scale parameter  $\sigma = \left( \int_{\mathbb{R}} |f(x)|^\alpha dx \right)^{1/\alpha}$  and  $T_f(\lambda)$  reduces to  $C(\alpha)\sigma^\alpha \lambda^{-\alpha}$ ; thus we recover the statement of Property 1.2.15, on page 16 in [4].
- Theorem 1.2 shows that when  $\|\alpha\|_{L^\infty(\mathbb{R})} < 2$ , then the distribution of the multistable random variable  $I(f)$  is heavy-tailed (see (2.25)).

## 2 Proof of the main result

The main goal of this section is to prove Theorem 1.2. To this end, we need to introduce some notations and to derive some preliminary result. We denote by  $f$  an arbitrary function of  $\mathcal{F}_\alpha^*$  and by  $I(f)$  the  $\alpha(x)$ -multistable random variable defined as the  $\alpha(x)$ -multistable stochastic integral of  $f$ . The characteristic function of  $I(f)$  is denoted by  $\Phi_f$ , recall that it is defined for all  $\theta \in \mathbb{R}$  as,

$$\Phi_f(\theta) = \mathbb{E}(e^{i\theta I(f)}). \quad (2.1)$$

Observe that, in view of (1.1), one has for all  $\theta \in \mathbb{R}$ ,

$$\Phi_f(\theta) = \exp \left\{ - \int_{\mathbb{R}} |\theta f(x)|^{\alpha(x)} dx \right\}. \quad (2.2)$$

As a consequence:

**Remark 2.1**  $\Phi_f$  is an even function which belongs to the Lebesgue space  $L^p(\mathbb{R})$ , for any arbitrary  $p \in (0, +\infty)$ , in particular it belongs to  $L^1(\mathbb{R})$ . Therefore, the distribution of  $I(f)$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ , moreover  $D_f$  the corresponding Randon-Nikodym derivative (i.e. the probability density function of  $I(f)$ ) is given for all  $x \in \mathbb{R}$ , by

$$D_f(x) = (2\pi)^{-1} \widehat{\Phi}(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-ix\theta} \Phi_f(\theta) d\theta, \quad (2.3)$$

which implies that  $D_f$  is a continuous, even and bounded function. Moreover, for all  $\theta \in \mathbb{R}$ ,

$$\Phi_f(\theta) = \widehat{D_f}(\theta) = \int_{\mathbb{R}} e^{-i\theta x} D_f(x) dx. \quad (2.4)$$

Notice that throughout this paper the Fourier transform of an arbitrary function  $h$  of  $L^1(\mathbb{R})$ , is defined, for all  $x \in \mathbb{R}$ , as  $\widehat{h}(x) = \int_{\mathbb{R}} e^{-ix\theta} h(\theta) d\theta$ .

PROOF OF REMARK 2.1: In view of (2.2), it is clear that  $\Phi_f$  is an even function, moreover by using the fact that for almost all  $x \in \mathbb{R}$ ,  $\alpha(x) \in [a, b] \subset (0, 2)$ , one has for all  $p \in (0, +\infty)$ ,

$$\begin{aligned} \int_{\mathbb{R}} |\Phi_f(\theta)|^p d\theta &= \int_{\mathbb{R}} \exp \left\{ -p \int_{\mathbb{R}} |\theta f(x)|^{\alpha(x)} dx \right\} d\theta \\ &\leq \int_{|\theta| \leq 1} \exp \left\{ -p |\theta|^b \int_{\mathbb{R}} |f(x)|^{\alpha(x)} dx \right\} d\theta + \int_{|\theta| > 1} \exp \left\{ -p |\theta|^a \int_{\mathbb{R}} |f(x)|^{\alpha(x)} dx \right\} d\theta < \infty. \end{aligned}$$

□

Let  $q$  be an arbitrary fixed real number strictly larger than 1, and let  $\varphi_q$  be an even real-valued  $C^\infty$  function whose Fourier transform  $\widehat{\varphi_q}$  is an even compactly supported  $C^5$  function with values in  $[0, 1]$  satisfying for all  $x \in \mathbb{R}$ ,

$$\widehat{\varphi_q}(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq \frac{1+q}{2}. \end{cases} \quad (2.5)$$

Observe that for all  $\gamma \in [0, 4)$ ,

$$\int_{\mathbb{R}} (1 + |\theta|)^\gamma \varphi_q(\theta) d\theta < \infty. \quad (2.6)$$

Also, observe that one has for all integer  $j \geq 0$ ,

$$\widehat{\varphi_q}(q^{-j}x) = 0 \text{ when } |x| \geq \left(\frac{1+q}{2}\right)q^j, \quad (2.7)$$

and

$$1 - \widehat{\varphi_q}(q^{-j}x) = 0 \text{ when } |x| \leq q^j. \quad (2.8)$$

For all  $\lambda \in [q, +\infty)$ , let  $j_0(\lambda, q) \geq 1$  be the unique integer such that

$$q^{j_0(\lambda, q)} \leq \lambda < q^{j_0(\lambda, q)+1}, \quad (2.9)$$

therefore, denoting by  $[\cdot]$  the integer part function, it follows that,

$$j_0(\lambda, q) = \left[ \frac{\log \lambda}{\log q} \right]. \quad (2.10)$$

**Lemma 2.2** *For all real number  $\xi \geq 1$ , let us set,*

$$\eta_f(\xi) = \int_{\mathbb{R}} \varphi_q(\theta) \left( 1 - \exp \left\{ - \int_{\mathbb{R}} |\xi^{-1} \theta f(x)|^{\alpha(x)} dx \right\} \right) d\theta. \quad (2.11)$$

*Then, for each  $\lambda \in [q, +\infty)$  one has,*

$$\eta_f(q^{j_0(\lambda, q)+1}) \leq \mathbb{P}(|I(f)| > \lambda) \leq \eta_f(q^{j_0(\lambda, q)-1}). \quad (2.12)$$

PROOF OF LEMMA 2.2: Using the fact that  $D_f$  is the probability density function of  $I(f)$ , one has that

$$\mathbb{P}(|I(f)| > \lambda) = \int_{|x| > \lambda} D_f(x) dx.$$

Therefore, it follows from (2.9) and (2.8) that,

$$\begin{aligned} \mathbb{P}(|I(f)| > \lambda) &\geq \int_{|x| \geq q^{j_0(\lambda, q)+1}} D_f(x) dx \geq \int_{|x| \geq q^{j_0(\lambda, q)+1}} (1 - \widehat{\varphi}_q(q^{-j_0(\lambda, q)-1}x)) D_f(x) dx \\ &= \int_{\mathbb{R}} (1 - \widehat{\varphi}_q(q^{-j_0(\lambda)-1}x)) D_f(x) dx; \end{aligned} \quad (2.13)$$

on the other hand, (2.9) and (2.7) imply that

$$\begin{aligned} \mathbb{P}(|I(f)| > \lambda) &\leq \int_{|x| \geq q^{j_0(\lambda, q)}} D_f(x) dx \\ &= \int_{|x| \geq q^{j_0(\lambda)}} \widehat{\varphi}_q(q^{-j_0(\lambda, q)+1}x) D_f(x) dx + \int_{|x| \geq q^{j_0(\lambda)}} (1 - \widehat{\varphi}_q(q^{-j_0(\lambda, q)+1}x)) D_f(x) dx \\ &\leq \int_{\mathbb{R}} (1 - \widehat{\varphi}_q(q^{-j_0(\lambda, q)+1}x)) D_f(x) dx. \end{aligned} \quad (2.14)$$

Let us now prove that for all real number  $\delta > 0$ , one has,

$$\int_{\mathbb{R}} (1 - \widehat{\varphi}_q(\delta x)) D_f(x) dx = \int_{\mathbb{R}} \varphi_q(\theta) (1 - \widehat{D}_f(\delta \theta)) d\theta. \quad (2.15)$$

In view of the fact that

$$\int_{\mathbb{R}} D_f(x) dx = 1 \text{ and } \int_{\mathbb{R}} \varphi_q(\theta) d\theta = \widehat{\varphi}_q(0) = 1,$$

it is sufficient to show that

$$\int_{\mathbb{R}} \widehat{\varphi}_q(\delta x) D_f(x) dx = \int_{\mathbb{R}} \varphi_q(\theta) \widehat{D}_f(\delta \theta) d\theta.$$

By using the definition of the Fourier transform of an  $L^1(\mathbb{R})$  function and Fubini Theorem, it follows that

$$\begin{aligned} \int_{\mathbb{R}} \widehat{\varphi}_q(\delta x) D_f(x) dx &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-i\delta x \theta} \varphi_q(\theta) d\theta \right) D_f(x) dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\delta x \theta} \varphi_q(\theta) D_f(x) d\theta dx = \int_{\mathbb{R}} \varphi_q(\theta) \left( \int_{\mathbb{R}} e^{-i\delta x \theta} D_f(x) dx \right) d\theta \\ &= \int_{\mathbb{R}} \varphi_q(\theta) \widehat{D}_f(\delta \theta) d\theta, \end{aligned}$$

thus one gets (2.15). Finally combining the latter relation with (2.13), (2.14), (2.4) and (2.2), one obtains the lemma.  $\square$

**Lemma 2.3** *For all real number  $u \geq 0$ , one has*

$$0 \leq u - 1 + e^{-u} \leq \frac{u^2}{2}. \quad (2.16)$$

PROOF OF LEMMA 2.3: Let  $\kappa_1$  and  $\kappa_2$  be the functions defined for all real number  $u \geq 0$  as,

$$\kappa_1(u) = u - 1 + e^{-u} \text{ and } \kappa_2(u) = \frac{u^2}{2} - \kappa_1(u).$$

One has  $\kappa_1(0) = 0$  and for all  $u \geq 0$ ,  $\kappa_1'(u) = 1 - e^{-u} \geq 0$ ; this implies that for each  $u \geq 0$ ,  $\kappa_1(u) \geq 0$ . One has  $\kappa_2(0) = 0$  and for every  $u \geq 0$ ,  $\kappa_2'(u) = \kappa_1(u) \geq 0$ ; this entails that for each  $u \geq 0$ ,  $\kappa_2(u) \geq 0$ .  $\square$

**Lemma 2.4** *Let  $h_q$  be the continuous function defined for all  $\gamma \in [a, b]$ , as,*

$$h_q(\gamma) = \int_{\mathbb{R}} |\theta|^\gamma \varphi_q(\theta) d\theta. \quad (2.17)$$

*Then, one has, for all  $\gamma \in [a, b]$ ,*

$$q^{-\gamma} h_q(\gamma) \leq C(\gamma) \leq q^\gamma h_q(\gamma), \quad (2.18)$$

*where  $C$  is the continuous and strictly positive function introduced in (1.3).*

PROOF OF LEMMA 2.4: First observe that in view of (2.6), the function  $h_q$  is well-defined and finite; moreover the dominated convergence theorem allows to prove that  $h_q$  is continuous on  $[a, b]$ . Let us now prove that (2.18) holds. Let  $\gamma$  be an arbitrary fixed real number belonging to the interval  $[a, b]$ . Assuming that  $\alpha(x) = \gamma$  for all  $x \in \mathbb{R}$ , then  $I(f)$  reduces to a symmetric  $\gamma$  stable random variable. Next it follows from (2.10), (2.11) and (2.12), that for all integer  $m \geq 1$ , one has,

$$\begin{aligned} & \int_{\mathbb{R}} \varphi_q(\theta) \left( 1 - \exp \left\{ - |q^{-m-1}\theta|^\gamma \int_{\mathbb{R}} |f(x)|^\gamma dx \right\} \right) d\theta \\ & \leq \mathbb{P}(|I(f)| > q^m) \\ & \leq \int_{\mathbb{R}} \varphi_q(\theta) \left( 1 - \exp \left\{ - |q^{-m+1}\theta|^\gamma \int_{\mathbb{R}} |f(x)|^\gamma dx \right\} \right) d\theta. \end{aligned} \quad (2.19)$$

On the other hand, Property 1.2.15 on page 16 in [4], implies that

$$\lim_{m \rightarrow +\infty} q^{m\gamma} \mathbb{P}(|I(f)| > q^m) = C(\gamma) \int_{\mathbb{R}} |f(x)|^\gamma dx. \quad (2.20)$$

Let us now show that

$$\lim_{m \rightarrow +\infty} q^{m\gamma} \int_{\mathbb{R}} \varphi_q(\theta) \left( 1 - \exp \left\{ - |q^{-m-1}\theta|^\gamma \int_{\mathbb{R}} |f(x)|^\gamma dx \right\} \right) d\theta = q^{-\gamma} h_q(\gamma) \int_{\mathbb{R}} |f(x)|^\gamma dx \quad (2.21)$$

and

$$\lim_{m \rightarrow +\infty} q^{m\gamma} \int_{\mathbb{R}} \varphi_q(\theta) \left( 1 - \exp \left\{ - |q^{-m+1}\theta|^\gamma \int_{\mathbb{R}} |f(x)|^\gamma dx \right\} \right) d\theta = q^\gamma h_q(\gamma) \int_{\mathbb{R}} |f(x)|^\gamma dx. \quad (2.22)$$

We will only prove (2.22) since (2.21) can be obtained similarly. To this end, we will use the dominated convergence theorem. It is clear that for all  $\theta \in \mathbb{R}$ ,

$$\lim_{m \rightarrow +\infty} q^{m\gamma} \varphi_q(\theta) \left( 1 - \exp \left\{ - |q^{-m+1}\theta|^\gamma \int_{\mathbb{R}} |f(x)|^\gamma dx \right\} \right) = q^\gamma |\theta|^\gamma \varphi_q(\theta) \int_{\mathbb{R}} |f(x)|^\gamma dx. \quad (2.23)$$

Moreover, it follows from the inequality in the left hand side of (2.16), that for all integer  $m \geq 2$  and real  $\theta$ ,

$$\begin{aligned} q^{m\gamma} |\varphi_q(\theta)| \left| 1 - \exp \left\{ - |q^{-m+1}\theta|^\gamma \int_{\mathbb{R}} |f(x)|^\gamma dx \right\} \right| &\leq q^{m\gamma} |\varphi_q(\theta)| |q^{-m+1}\theta|^\gamma \int_{\mathbb{R}} |f(x)|^\gamma dx \\ &= q^\gamma |\theta|^\gamma |\varphi_q(\theta)| \int_{\mathbb{R}} |f(x)|^\gamma dx. \end{aligned} \quad (2.24)$$

In view of (2.23) and (2.24), we are allowed to apply the dominated convergence theorem, and thus we obtain (2.22). Finally, putting together (2.19), (2.20), (2.21) and (2.22), one gets (2.18).  $\square$

Let us now give some useful properties of the function  $T_f$  defined in (1.4); easy computations allow to obtain the following two remarks.

**Remark 2.5** *One has for all real number  $\xi \geq 1$ ,*

$$\xi^{-b} T_f(1) \leq T_f(\xi) \leq \xi^{-a} T_f(1). \quad (2.25)$$

**Remark 2.6** *For each real numbers  $\delta > 0$  and  $\lambda > 0$ ,*

(i) *when  $\delta \in (0, 1]$ , one has*

$$\delta^{-a} T_f(\lambda) \leq T_f(\delta\lambda) \leq \delta^{-b} T_f(\lambda), \quad (2.26)$$

(ii) *when  $\delta > 1$ , one has*

$$\delta^{-b} T_f(\lambda) \leq T_f(\delta\lambda) \leq \delta^{-a} T_f(\lambda). \quad (2.27)$$

**Lemma 2.7** *For every real number  $\xi \geq 1$ , we set,*

$$\rho_f(\xi) = \int_{\mathbb{R}} |\varphi_q(\theta)| \left| \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx - 1 + \exp \left\{ - \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx \right\} \right| d\theta. \quad (2.28)$$

*Then, there is a constant  $c(q) > 0$ , only depending on  $b$  and  $q$  such that for all  $\lambda \geq q$ ,*

$$\sup_{f \in S_\alpha} \frac{\rho_f(q^{j_0(\lambda, q)+1})}{T_f(\lambda)} \leq c(q) \lambda^{-a} \quad (2.29)$$

and

$$\sup_{f \in \mathcal{S}_\alpha} \frac{\rho_f(q^{j_0(\lambda, q)-1})}{T_f(\lambda)} \leq c(q)\lambda^{-a}; \quad (2.30)$$

recall that  $\mathcal{S}_\alpha$  is the unit sphere of  $\mathcal{F}_\alpha$  and that the strictly positive integer  $j_0(\lambda, q)$  has been introduced in (2.10).

PROOF OF LEMMA 2.7: It follows from (2.28) and from the right hand side inequality in (2.16) (in which one takes  $u = \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx$ ), that for all real number  $\xi \in [1, +\infty)$ ,

$$\rho_f(\xi) \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx \right)^2 |\varphi_q(\theta)| d\theta. \quad (2.31)$$

Then noticing that

$$\left( \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx \right)^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi^{-1}\theta f(x_1)|^{\alpha(x_1)} |\xi^{-1}\theta f(x_2)|^{\alpha(x_2)} dx_1 dx_2,$$

and using Fubini-Tonelli Theorem, one gets that

$$\begin{aligned} & \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx \right)^2 |\varphi_q(\theta)| d\theta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi^{-1}f(x_2)|^{\alpha(x_1)} |\xi^{-1}f(x_2)|^{\alpha(x_2)} \left( \int_{\mathbb{R}} |\theta|^{\alpha(x_1)+\alpha(x_2)} |\varphi_q(\theta)| d\theta \right) dx_1 dx_2 \\ &\leq c_1 \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi^{-1}f(x_2)|^{\alpha(x_1)} |\xi^{-1}f(x_2)|^{\alpha(x_2)} dx_1 dx_2 \\ &= c_1 \left( \int_{\mathbb{R}} |\xi^{-1}f(x)|^{\alpha(x)} dx \right)^2, \end{aligned} \quad (2.32)$$

where, in view of (2.6),

$$c_1 = \int_{\mathbb{R}} (1 + |\theta|)^{2b} |\varphi_q(\theta)| d\theta,$$

is a finite constant. Now let us set

$$c_2 = \max_{y \in [a, b]} (C(y))^{-1};$$

observe that the latter constant is finite since  $C$  (see (1.3)) is a strictly positive continuous function on  $[a, b]$ . Then using (1.4), one has that, for all real number  $\xi \geq 1$ ,

$$\int_{\mathbb{R}} |\xi^{-1}f(x)|^{\alpha(x)} dx \leq c_2 T_f(\xi). \quad (2.33)$$

Next, combining (2.31) with (2.32) and (2.33), it follows that,

$$\rho_f(\xi) \leq c_3 (T_f(\xi))^2, \quad (2.34)$$



where  $c_3 = c_1 c_2^2$ . Next using (2.34), (2.9) and the fact that  $T_f$  is a nonincreasing function, one obtains that for all real number  $\lambda \geq q$ ,

$$0 < \frac{\rho_f(q^{j_0(\lambda, q)+1})}{T_f(\lambda)} \leq c_3 \frac{(T_f(q^{j_0(\lambda, q)+1}))^2}{T_f(\lambda)} \leq c_3 T_f(\lambda) \quad (2.35)$$

and

$$0 < \frac{\rho_f(q^{j_0(\lambda, q)-1})}{T_f(\lambda)} \leq c_3 \frac{(T_f(q^{j_0(\lambda, q)-1}))^2}{T_f(\lambda)} \leq c_3 \frac{(T_f(q^{-2}\lambda))^2}{T_f(\lambda)} \leq q^{4b} c_3 T_f(\lambda), \quad (2.36)$$

where the latter inequality follows from (2.26). Next setting

$$c(q) = c_3 \left( 1 + q^{4b} \max_{y \in [a, b]} C(y) \right),$$

and observing that for all  $f \in \mathcal{S}_\alpha$ ,

$$T_f(1) \leq \left( \max_{y \in [a, b]} C(y) \right) \int_{\mathbb{R}} |f(x)|^{\alpha(x)} dx = \left( \max_{y \in [a, b]} C(y) \right),$$

then (2.35), (2.36) and Remark 2.5 imply that (2.29) and (2.30) hold.  $\square$

**Lemma 2.8** *For every real number  $\xi \geq 1$ , we set,*

$$\tau_f(\xi) = \int_{\mathbb{R}} \varphi_q(\theta) \left( \int_{\mathbb{R}} |\xi^{-1} \theta f(x)|^{\alpha(x)} dx \right) d\theta. \quad (2.37)$$

*Then, one has, for all real number  $\xi \geq q$ ,*

$$T_f(q\xi) \leq \tau_f(\xi) \leq T_f(q^{-1}\xi); \quad (2.38)$$

*moreover*

$$q^{-2b} \leq \inf_{\lambda \in [q, +\infty)} \frac{\tau_f(q^{j_0(\lambda, q)+1})}{T_f(\lambda)} \leq \sup_{\lambda \in [q, +\infty)} \frac{\tau_f(q^{j_0(\lambda, q)-1})}{T_f(\lambda)} \leq q^{3b}. \quad (2.39)$$

**PROOF OF LEMMA 2.8:** First observe that (2.6) and the fact that  $f \in \mathcal{F}_\alpha$ , imply that for all real number  $\xi \geq 1$ ,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi_q(\theta)| |\xi^{-1} \theta f(x)|^{\alpha(x)} dx d\theta \leq \left( \int_{\mathbb{R}} (1 + |\theta|)^b |\varphi_q(\theta)| d\theta \right) \times \left( \int_{\mathbb{R}} |f(x)|^{\alpha(x)} dx \right) < \infty.$$

Therefore, we are allowed to use Fubini Theorem and we obtain that, for all real number  $\xi \geq 1$ ,

$$\tau_f(\xi) = \int_{\mathbb{R}} |\xi^{-1} f(x)|^{\alpha(x)} h_q(\alpha(x)) dx. \quad (2.40)$$

where the function  $h_q$  has been introduced in (2.17). Next it follows from (2.18) that, for all real number  $\xi \geq 1$ ,

$$\int_{\mathbb{R}} |\xi^{-1} f(x)|^{\alpha(x)} q^{-\alpha(x)} C(\alpha(x)) dx \leq \tau_f(\xi) \leq \int_{\mathbb{R}} |\xi^{-1} f(x)|^{\alpha(x)} q^{\alpha(x)} C(\alpha(x)) dx;$$

thus, in view of (1.4), one gets (2.38). Let us now prove that the first inequality in (2.39) holds. Assume that the real  $\lambda \geq q$  is arbitrary. It follows from the first inequality in (2.38), (2.9), the fact that  $T_f$  is a nonincreasing function and the first inequality in (2.27) (in which one takes  $\delta = q^2$ ), that

$$\frac{\tau_f(q^{j_0(\lambda,q)+1})}{T_f(\lambda)} \geq \frac{T_f(q^{j_0(\lambda,q)+2})}{T_f(\lambda)} \geq \frac{T_f(q^2\lambda)}{T_f(\lambda)} \geq q^{-2b},$$

thus we obtain the first inequality in (2.39). Next observe that (2.40) and the fact that  $h_q$  is a (strictly) positive function (this is a straightforward consequence of (2.18)) imply that  $\tau_f$  is a nonincreasing function, which in turn entails that for all real number  $\lambda \geq q$  one has  $\tau_f(q^{j_0(\lambda,q)+1}) \leq \tau_f(q^{j_0(\lambda,q)-1})$  and, as consequence, that the second inequality in (2.39) is satisfied. Finally, let us show that the last inequality in (2.39) holds. It follows from the second inequality in (2.38), (2.9), the fact that  $T_f$  is a nonincreasing function and the second inequality in (2.26) (in which one takes  $\delta = q^{-3}$ ), that

$$\frac{\tau_f(q^{j_0(\lambda,q)-1})}{T_f(\lambda)} \leq \frac{T_f(q^{j_0(\lambda,q)-2})}{T_f(\lambda)} \leq \frac{T_f(q^{-3}\lambda)}{T_f(\lambda)} \leq q^{3b},$$

thus we obtain the last inequality in (2.39).  $\square$

**Lemma 2.9** *Let  $\eta_f$  be the function introduced in (2.11) and let  $c(q) > 0$  be the constant introduced in Lemma 2.7. Then one has for all  $f \in \mathcal{S}_\alpha$ , and for all real number  $\lambda \geq q$*

$$q^{-2b} - c(q)\lambda^{-a} \leq \frac{\eta_f(q^{j_0(\lambda,q)+1})}{T_f(\lambda)} \leq \frac{\eta_f(q^{j_0(\lambda,q)-1})}{T_f(\lambda)} \leq q^{3b} + c(q)\lambda^{-a}. \quad (2.41)$$

PROOF OF LEMMA 2.9: In view of (2.11), for all real number  $\xi \geq 1$ , one can write,

$$\begin{aligned} \eta_f(\xi) &= \int_{\mathbb{R}} \varphi_q(\theta) \left( \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx \right) d\theta \\ &\quad + \int_{\mathbb{R}} \varphi_q(\theta) \left( 1 - \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx - \exp \left\{ - \int_{\mathbb{R}} |\xi^{-1}\theta f(x)|^{\alpha(x)} dx \right\} \right) d\theta; \end{aligned}$$

thus, it follows from the triangle inequality (2.37) and (2.28), that for each real number  $\xi \geq 1$ , one has,

$$\tau_f(\xi) - \rho_f(\xi) \leq \eta_f(\xi) \leq \tau_f(\xi) + \rho_f(\xi). \quad (2.42)$$

Let us now prove that the first inequality in (2.41) holds. Using the first inequality in (2.42) as well as the first inequality in (2.39), one has for all real number  $\lambda \geq q$ ,

$$q^{-2b} - \frac{\rho_f(q^{j_0(\lambda,q)+1})}{T_f(\lambda)} \leq \frac{\eta_f(q^{j_0(\lambda,q)+1})}{T_f(\lambda)},$$

then the first inequality in (2.41) results from (2.29). Next, observe that (2.12) clearly implies that the second inequality in (2.41) is satisfied. Finally, let us show that the last inequality in

(2.41) holds. Using the second inequality in (2.42) as well as the second inequality in (2.39), one has for all real number  $\lambda \geq q$ ,

$$\frac{\eta_f(q^{j_0(\lambda,q)-1})}{T_f(\lambda)} \leq q^{3b} + \frac{\rho_f(q^{j_0(\lambda,q)-1})}{T_f(\lambda)};$$

then the last inequality in (2.41) results from (2.30).  $\square$

Now we are in position to prove Theorem 1.2.

PROOF OF THEOREM 1.2: Let  $c(q)$  be the constant introduced in Lemma 2.7. Using (2.12) and (2.41), one has for all  $f \in \mathcal{S}_\alpha$  and all real number  $\lambda \geq q$ ,

$$q^{-2b} - c(q)\lambda^{-a} \leq \frac{\eta_f(q^{j_0(\lambda,q)+1})}{T_f(\lambda)} \leq \frac{\mathbb{P}(|I(f)| > \lambda)}{T_f(\lambda)} \leq \frac{\eta_f(q^{j_0(\lambda,q)-1})}{T_f(\lambda)} \leq c(q)\lambda^{-a} + q^{3b},$$

which implies that,

$$q^{-2b} - 1 - c(q)\lambda^{-a} \leq \frac{\mathbb{P}(|I(f)| > \lambda)}{T_f(\lambda)} - 1 \leq c(q)\lambda^{-a} + q^{3b} - 1$$

and, as a consequence, that

$$\sup_{f \in \mathcal{S}_\alpha} \left| \frac{\mathbb{P}(|I(f)| > \lambda)}{T_f(\lambda)} - 1 \right| \leq |q^{-2b} - 1| + |q^{3b} - 1| + c(q)\lambda^{-a}.$$

One has therefore,

$$\limsup_{\lambda \rightarrow +\infty} \left\{ \sup_{f \in \mathcal{S}_\alpha} \left| \frac{\mathbb{P}(|I(f)| > \lambda)}{T_f(\lambda)} - 1 \right| \right\} \leq |q^{-2b} - 1| + |q^{3b} - 1|.$$

Finally letting  $q > 1$  goes to 1, one obtains the theorem.  $\square$

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